An introduction to Coxeter groups and the properties of their weak order

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1. Introduction
2. Coxeter Group Types
3. Posets
4. Sperner Property
5. Acknowledgements

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Introduction to Groups

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Example: $(\mathbb{Z}, +)$
Identity Element: 0
Inverse Element: $\forall\ a \in \mathbb{Z},\ a^{-1} = -a$

Generators

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- For example if $S = \{x, y\}$, then $G = \{w(x, y, x^{-1}, y^{-1})\}$

Coxeter Groups

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Definition

We consider a set S. A Coxeter matrix M with elements from $\{1, 2, \ldots, \infty\}$ satisfies the properties $M_{s,s'} = M_{s',s}$, and $M_{s,s'} = 1 \iff s = s'$

Coxeter Matrices Examples

- These extend to Coxeter Graphs where if $M_{i,j} = 2$, there exists no edge between i and j, and anything greater than 3 indicates an edge with a weight

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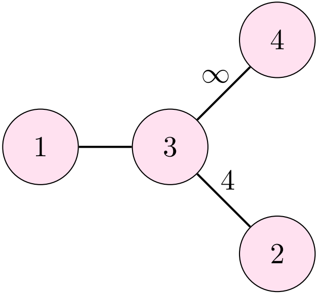
$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 3 & 4 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix}$$



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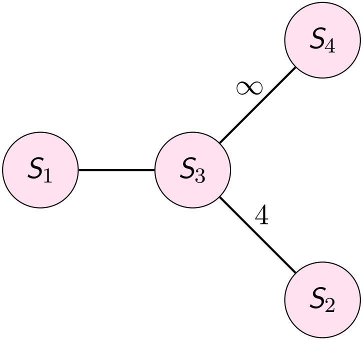
Understanding Coxeter Groups

- A Coxeter matrix determines a group G with S as a set of generators, where $(ss')^{M_{s,s'}} = e$
- This means that we impose the relation $s^2 = e$
- Example:

$$\begin{aligned} & x * y * y^2 * x^3 * y^3 * y * y^2 && (1) \\ &= x * y * x * x^2 * y * y^2 * y && (2) \\ &= x * y * x * y * y && (3) \\ &= x * y * x * y^2 && (4) \\ &= x * y * x && (5) \end{aligned}$$

Coxeter Example

- G is the Coxeter group and S is the set of Coxeter generators
- We can think our last last example of a graph of 4 generators, S_1, S_2, S_3 and S_4, which all have the property $S_i^2 = e$



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3 Posets

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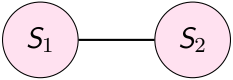
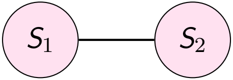


Generators



- We consider the transpositions $S_1 = (1\ \ 2)$ and $S_2 = (2\ \ 3)$
- We begin with 123

Generating A_2

Recall that $S_1 = (1 \;\; 2)$ and $S_2 = (2 \;\; 3)$

$$123$$

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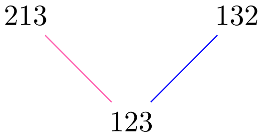

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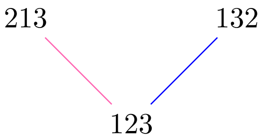

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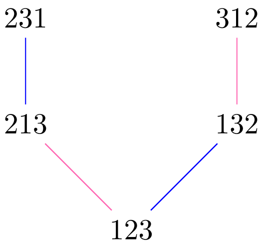

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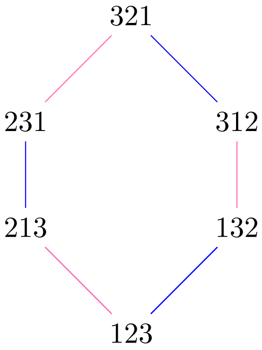
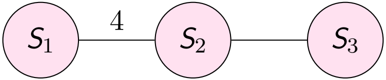


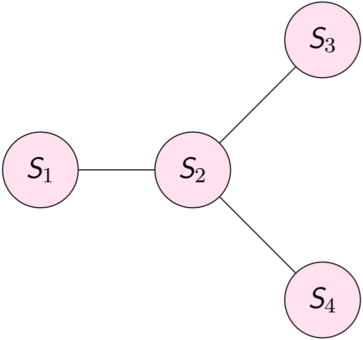
Type B and D Coxeter Groups

- We also have type B Coxeter groups
- For example, B_3 has the Coxeter matrix $\begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$



Type B and D Coxeter Groups Continued

- Then there are type D Coxeter groups
- D_4 has the Coxeter matrix $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 3 \\ 2 & 3 & 1 & 2 \\ 2 & 3 & 2 & 1 \end{pmatrix}$



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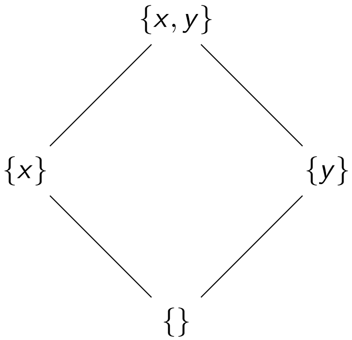
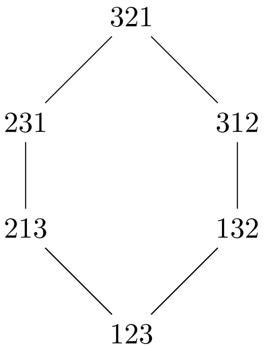
Introduction to Posets

- Posets stand for Partially Ordered Sets
- Posets have a set P with a partial order relation $\leq$
- Posets are transitive, reflexive and antisymmetric



Length Function



Here we have a poset, where we can compare our elements using a length function $\ell(w)$, where we consider the shortest number of transpositions from 123 to obtain our new word

Ex: $\ell(231) = 2$

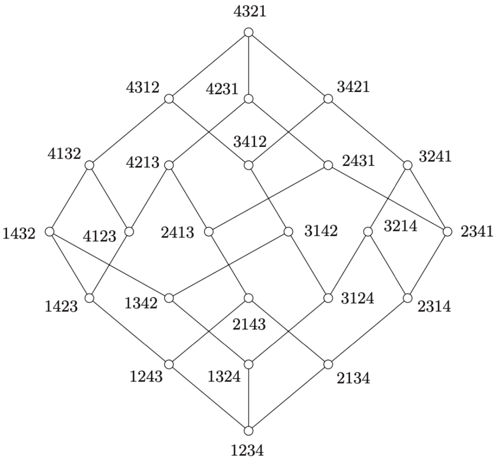
Weak Order of Coxeter Groups

Definition

The right weak order of a Coxeter group (G, S) states for $u, w \in G$, if $w = us_1s_2 \dots s_k$, for some $s_i \in S$ such that $\ell(us_1s_2 \dots s_k) = \ell(u) + i,\ 0 \leq i \leq k$, then $u \leq w$

Example of Weak Order

Here we can see the weak order of the Coxeter group A_3



Poset Antichains

- An antichain is a subset of nodes in our poset such that all of the nodes are incomparable to each other

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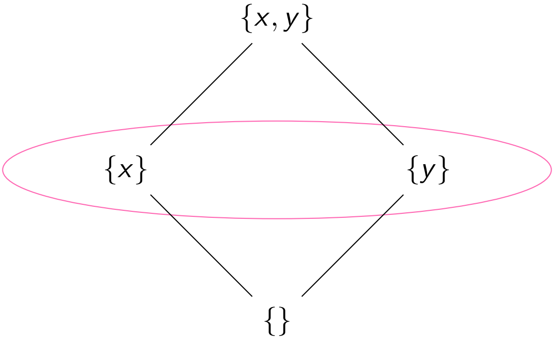
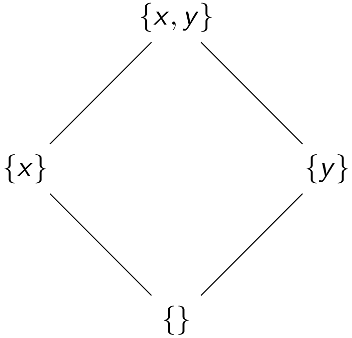


Rank of a Poset

- A chain is a set of nodes such that all the nodes are comparable
- A ranked poset has maximal chains of equal length. A maximal chain is a chain such that no superset is also a chain



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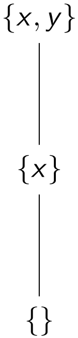
- The Sperner property describes posets where the size of the largest antichain is less than or equal to the largest rank



Type A Coxeter Groups and the Sperner Property

Theorem (Gaetz and Gao)

The weak order of type A Coxeter groups are strongly Sperner

Insert... Conjecture

A COMBINATORIAL $\mathfrak{sl}_2$-ACTION AND THE SPERNER PROPERTY FOR THE WEAK ORDER

CHRISTIAN GAETZ AND YIBO GAO

ABSTRACT. We construct a simple combinatorially-defined representation of $\mathfrak{sl}_2$ which respects the order structure of the weak order on the symmetric group. This is used to prove that the weak order has the strong Sperner property, and is therefore a Peck poset, solving a problem raised by Björner (1984); a positive answer to this question had been conjectured by Stanley (2017).

Conjecture 3.1

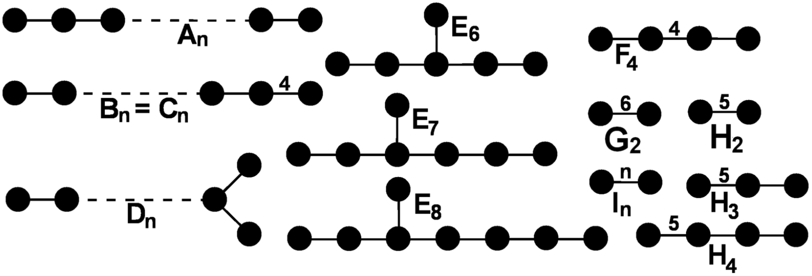
Conjecture 3.1. *The weak order on any finite Coxeter group strongly Sperner.*

An easy argument proves the Conjecture for the dihedral groups, and computer checks have also verified it for all Coxeter groups of rank at most four.

While this is recognized as an open problem, this paper conjectures that all finite Coxeter groups are strongly Sperner. I've been given the project of disproving this conjecture . . .

Why Do We Care?

- There are a lot of more complicated finite Coxeter groups, many of which are very difficult to study as they get significantly more complicated
- Discovering more properties helps us learn more about the complicated cases



My Research This Summer

- Using deep cross-entropy methods
- This approach comes from a paper Adam Zsolt Wagner

Algorithm 1: The deep cross-entropy method

```
Initialize a neural network;
while the best construction found is not a counterexample do
    for i ← 1 to N do
        w ← empty string;
        while not terminal do
            Input w into the neural net to get a probability distribution F on the next letter;
            Sample next letter x according to F;
            w ← w + x;
        end
    end
    Evaluate the score of each construction;
    Sort the constructions according to their score;
    Throw away all but the top y percentage of the constructions;
    for all remaining constructions do
        for all (observation, issued action) pairs in the construction do
            Adjust the weights of the neural net slightly to minimize the cross-entropy loss
              between issued action and the corresponding predicted action probability;
        end
    end
    Keep the top x percentage of constructions for the next iteration, throw away the rest;
end
```

Using Sage

- Using SageMath, we have found that D_5 and E_6 are Sperner, but will keep using similar techniques for E_7, E_8 and unions of antichains

```
sage: WeylGroup(["D", 4]).weak_poset().width()
30
sage: WeylGroup(['B', 4]).weak_poset().width()
46
sage: WeylGroup(['B', 5]).weak_poset().width()
340
sage: WeylGroup(['D', 5]).weak_poset().width()
212
sage: Hello URA Seminar!
```

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References:

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Coxeter–dynkin diagram. Academic Dictionaries and Encyclopedias. (n.d.). Retrieved July 16, 2022, from https://en-academic.com/dic.nsf/enwiki/11585259

Gaetz, C., & Gao, Y. (2019). A combinatorial $\mathfrak{sl}_2$-action and the Sperner property for the weak order. *Proceedings of the American Mathematical Society*, *148*(1), 1–7. https://doi.org/10.1090/proc/14655

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Thank you!