An introduction to Coxeter groups and the properties of their weak order

Kimia Shaban and (Yohan Song, Willie Lei)
URA Supervisor: Oliver Pechenik

University of Waterloo
Department of Combinatorics and Optimization

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Understanding Reflections

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- In higher dimensions $(\mathbb{R}^n)$, a reflection will send $\alpha \in \mathbb{R}^n$ to its negative, while the hyperplane H_α orthogonal to α is fixed pointwise.

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  - The dihedral group of order $2n$, written (D_{2n}), has the form $D_{2n} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = 1 \rangle$, are also generated by reflections s_1 and s_2 with the relation $(s_1 s_2)^n = 1$.

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Definition

We consider a set S. A Coxeter matrix M with elements from $\{1, 2, \ldots, \infty\}$ satisfies the properties $M_{s,s'} = M_{s',s}$, and $M_{s,s'} = 1 \iff s = s'$

Coxeter Matrices Examples

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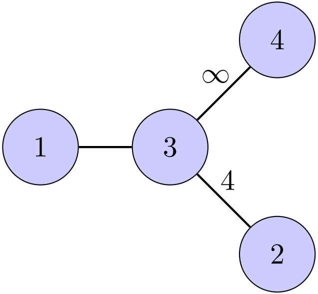
$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 3 & 4 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix}$$



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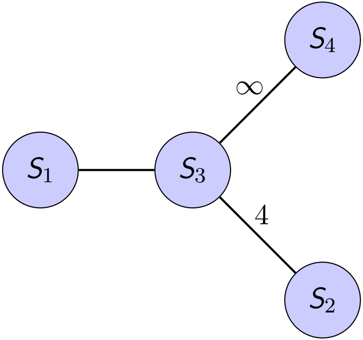
Understanding Coxeter Groups

- A Coxeter matrix determines a group G with S as a set of generators, where $(ss')^{M_{s,s'}} = e$
- This means that we impose the relation $s^2 = e$
- G is the Coxeter group and S is the set of Coxeter generators
- We can think our last last example of a graph of 4 generators, S_1, S_2, S_3 and S_4, which all have the property $S_i^2 = 1$



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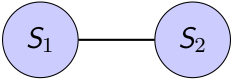
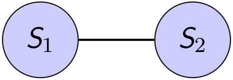


Generators



- We consider the transpositions $S_1 = (1\ \ 2)$ and $S_2 = (2\ \ 3)$
- We begin with 123

Generating A_2

Recall that $S_1 = (1\ \ 2)$ and $S_2 = (2\ \ 3)$

$$123$$

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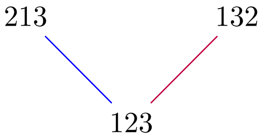

Generating A_2

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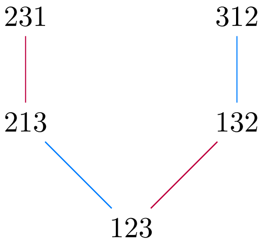

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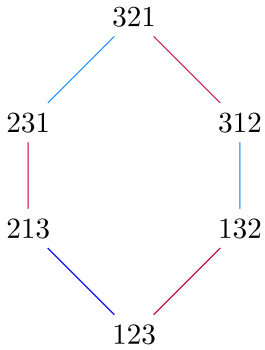
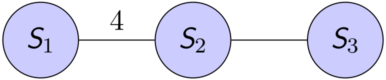


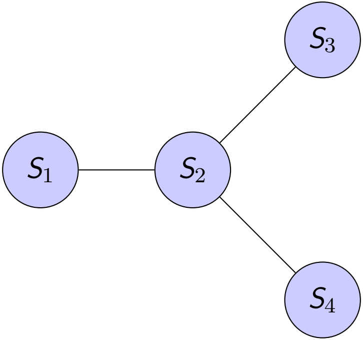
Type B and D Coxeter Groups

- We also have type B Coxeter groups
- For example, B_3 has the Coxeter matrix $\begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$



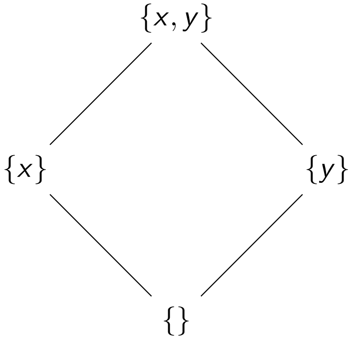
Type B and D Coxeter Groups Continued

- Then there are type D Coxeter groups
- D_4 has the Coxeter matrix $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 3 \\ 2 & 3 & 1 & 2 \\ 2 & 3 & 2 & 1 \end{pmatrix}$



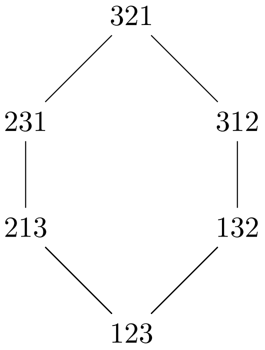
Introduction to Posets

- Posets stand for Partially Ordered Sets
- Posets have a set P with a partial order relation $\leq$
- Posets are transitive, reflexive and antisymmetric



Length Function

We define the length of an element in our Coxeter group as the smallest number of reflections used to generate it.



Here we have a poset, where we can compare our elements using a length function $\ell(w)$, where we consider the shortest number of transpositions from 123 to obtain our new word

Ex: $\ell(231) = 2$

Strong/Weak Order of Coxeter Groups

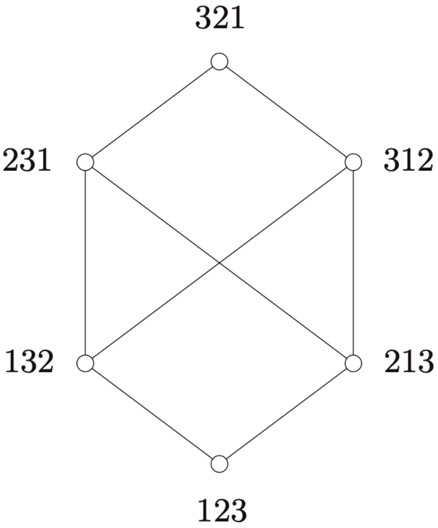
Definition

The strong order (in S_n), states that for $w \in S_n$ we state that $w \leq wt_{ij}$ if $\ell(wt_{ij}) = \ell(w) + 1$ where $t_{ij} = (i\ j)$

Definition

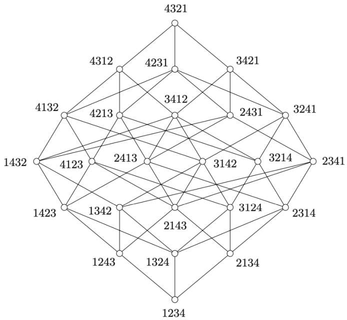
The right weak order (in S_n) of a Coxeter group (G, S) states that for $w \in S_n$ we state that $w \leq ws_i$ if $\ell(ws_i) = \ell(w) + 1$ where $s_i = (i\ i+1)$

Strong Order of A_2

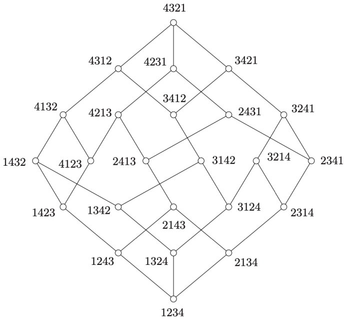


Examples of Strong/Weak ordefr

We observe the strong and weak order of the Coxeter group A_3



(a) Strong Order



(b) Weak Order

Poset Antichains

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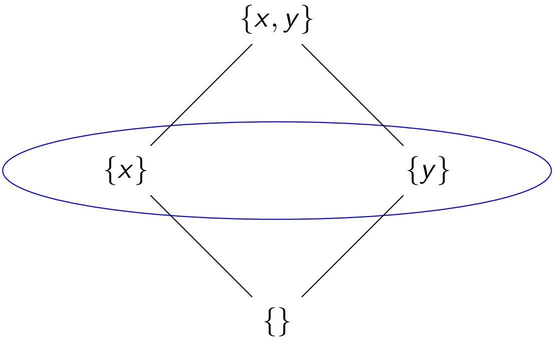
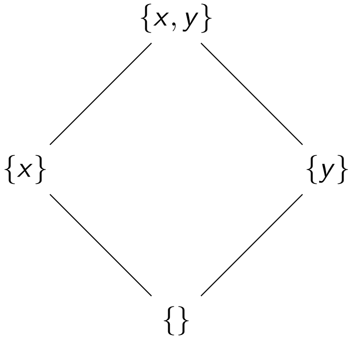
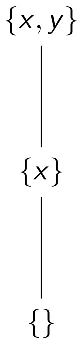


Rank of a Poset

- A chain is a set of nodes such that all the nodes are comparable
- A ranked poset has maximal chains of equal length. A maximal chain is a chain such that no superset is also a chain



- The Sperner property describes posets where the size of the largest antichain is less than or equal to the rank of the poset



Type A Coxeter Groups and the Sperner Property

Definition

A poset is k Sperner, if no union of k antichains is larger than the union of its largest k ranks. A poset is if it is k-Sperner for all $k \in \mathbb{N}$.

Theorem (Gaetz and Gao)

The weak order of type A Coxeter groups are strongly Sperner

Insert... Conjecture

A COMBINATORIAL $\mathfrak{sl}_2$-ACTION AND THE SPERNER PROPERTY FOR THE WEAK ORDER

CHRISTIAN GAETZ AND YIBO GAO

ABSTRACT. We construct a simple combinatorially-defined representation of $\mathfrak{sl}_2$ which respects the order structure of the weak order on the symmetric group. This is used to prove that the weak order has the strong Sperner property, and is therefore a Peck poset, solving a problem raised by Björner (1984); a positive answer to this question had been conjectured by Stanley (2017).

Conjecture 3.1

Conjecture 3.1. *The weak order on any finite Coxeter group strongly Sperner.*

An easy argument proves the Conjecture for the dihedral groups, and computer checks have also verified it for all Coxeter groups of rank at most four.

While this is recognized as an open problem, this paper conjectures that all finite Coxeter groups are strongly Sperner.

Focus of Research This Summer

- Using deep cross-entropy methods
- This approach comes from a paper Adam Zsolt Wagner

Algorithm 1: The deep cross-entropy method

```
Initialize a neural network;
while the best construction found is not a counterexample do
    for i ← 1 to N do
        w ← empty string;
        while not terminal do
            Input w into the neural net to get a probability distribution F on the next letter;
            Sample next letter x according to F;
            w ← w + x;
        end
    end
    Evaluate the score of each construction;
    Sort the constructions according to their score;
    Throw away all but the top y percentage of the constructions;
    for all remaining constructions do
        for all (observation, issued action) pairs in the construction do
            Adjust the weights of the neural net slightly to minimize the cross-entropy loss
              between issued action and the corresponding predicted action probability;
        end
    end
    Keep the top x percentage of constructions for the next iteration, throw away the rest;
end
```

Scoring Antichains

- In order to use the machine learning algorithm on Coxeter groups, we had to score subsets of elements created by the elements of our Coxeter groups.
- This involved creating states generated by comparing the elements in our "supposed" antichain, and maximizing the highest possible score to prevent comparable elements.

Using Sage

- Using SageMath, we have found that D_5 and E_6 are Sperner, but will keep using similar techniques for E_7, E_8 and unions of antichains

```
sage: WeylGroup(["D", 4]).weak_poset().width()
30
sage: WeylGroup(["B", 4]).weak_poset().width()
46
sage: WeylGroup(["B", 5]).weak_poset().width()
340
sage: WeylGroup(["D", 5]).weak_poset().width()
212
sage: Hello Algebraic Combinatorics & Enumeration Seminar!
```

Bipartite Matching Algorithm

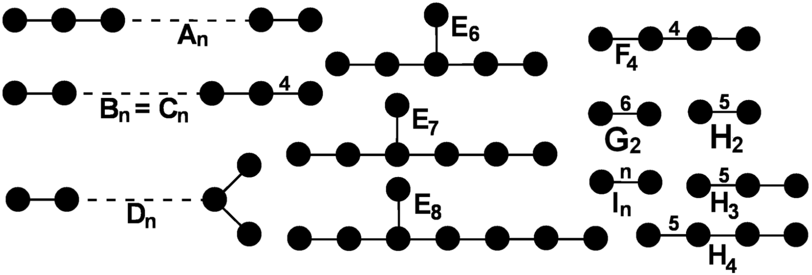
- The width of the poset returned from sage is equal to the number of chains, which by Dilworth's theorem states is equivalent to the largest antichain.
- Sage creates a bipartite graph, to create a matching to construct the union of chains to obtain the poset width.

Dilworth's Theorem

In any finite partially ordered set, the largest antichain has the same size as the smallest chain decomposition.

Interest in Coxeter groups

- There are a lot of more complicated finite Coxeter groups, many of which are very difficult to study as they get significantly more complicated
- Discovering more properties helps us learn more about the complicated cases



Applications of Coxeter groups

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- Lie algebra's have applications in quantum mechanics related to particle spin, and in particle physics.
- Weyl groups play a role in understanding both structure theory and representation theory.

Advancements

- Determining if a poset has Dilworth's number k can be done in $O(k^2 n^2)$ time.

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- In the classical case, the bipartite matching algorithm is $O(n^3)$.
- The maximal bipartite matching algorithm can be run in $O(n\sqrt{m+n}logn)$ using a Quantum algorithm, in a graph with n vertices and m edges.

References:

Björner, A., & Brent, F. (2000). *Combinatorics of Coxeter groups*. Scholars Portal.

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Thank you!